System Theory Exam 2012

1. (a) If

$$L = p(t)\frac{d^2}{dt^2} + q(t)\frac{d}{dt} + r(t)$$

and we use the standard inner L^2-inner product (real spaces only),

$$(x, y) := \int_a^b x(t)y(t)\, dt$$

demonstrate that the *formal adjoint* of operator L is given by the formula,

$$L^* = p\frac{d^2}{dt^2} + (2p' - q)\frac{d}{dt} + (p'' - q' + r)$$

(b) Determine sufficient conditions on coefficients p, q, r (the more general, the better) for the operator L to be self-adjoint.

(c) Provide sample boundary conditions for operator L under which the operator is self-adjoint.

2. (a) Discuss shortly the relation between Jordan Decomposition Theorem for a matrix $\mathbf{A}$ and solution of system of ODEs (with constant coefficients) of the form

$$\dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x}$$

(b) Find the general solution for the system

$$\begin{cases} 2\dot{x} &= 5x + y \\ 2\dot{y} &= -x + 3y \end{cases}$$

3. Find and identify (max, min, saddle) the point that optimizes the performance index

$$J = x^2 + u^2$$

subject to the inequality constraint

$$x - 2u \geq 1.$$

Use a slack variable and a Lagrange multiplier.

4. a. Find the control history $u(t)$ that minimizes the time ($J = t_f$) to drive the system

$$\dot{x}_1 = \cos\theta$$

$$\dot{x}_2 = \sin\theta$$

from the initial conditions

$$t_0 = 0, \quad x_{1_0} = 0, \quad x_{2_0} = 0$$

to the final conditions

$$x_{1_f} = 1, \quad x_{2_f} = 0.$$

b. Apply the Legendre-Clebsch condition and the Weierstrass condition to verify that the solution can be a minimum.

c. In words, what is the conjugate point condition? What does it mean if a conjugate point exists within the interval of integration?

5. (a) Two scalar observations are taken of the scalar parameter x:

$$y_1 = x + \epsilon_1 \qquad y_2 = x + \epsilon_2$$

where ϵ_1 and ϵ_2 are independent, zero-mean random variables. Thus,

$$E[\epsilon_1] = 0 \qquad E[\epsilon_2] = 0 \qquad E[\epsilon_1 \epsilon_2] = 0$$

It is known that the variance of ϵ_2 is twice the variance of ϵ_1. Determine the linear unbiased minimum variance estimate of x. (This is equivalent to the Maximum Likelihood estimate of x when ϵ_1 and ϵ_2 are viewed as Gaussian.)

(b) Calculate the variance of the estimate.

6. For each of the following cases, assume you want to calculate an estimate at $t_0 = 0$. For each case, say whether the system is observable or not observable, and explain your answer.

(a) A car moves with constant speed v on the x-axis. A tracking station is located on the x-axis at position x_s. Range data is taken from the station to the car. Estimate x_0, v, and x_s, where $x_0 = x(t_0)$.

(b) Same as (a), except both range and range-rate date are taken.

(c) A car moves with constant acceleration a on the x-axis. A tracking station is located on the x-axis at position x_s, which is known. Range data is taken from the station to the car. Estimate x_0, v_0, and a, where $v_0 = v(t_0)$.

(d) Same as (c), except range-rate data is taken instead of range data.

(e) A car moves with constant speed v along a circular track with radius r in the xy plane. The track is centered at $x = 0$, $y = 0$. Thus, the z axis passes through the center of the track and is perpendicular to the track. A tracking station is located at height h on the z-axis. Range date is taken from the station to the car. Assume v is known. Estimate r and h.

(f) Same as (e), except h is known, and estimate r.